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Necessary conditions for stability of a general class of compressible vortex flows are obtained using the method of generalized progressing wave expansion. The vortex motion and its density are assumed to vary in both the radial and axial directions. These conditions, representing a state of statically stable distribution for the steady flow, require that the flow be stable in the centrifugal force field created by the rotation of the fluids, in the gravitational force field arised

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20. ABSTRACT (Continued)

from the density variation in the vertical direction, and in the pressure field constraining the density stratification in both directions. These results also suggest the likely behavior of the Richardson criterion for stability of the flow yet to be derived.

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ON THE STABILITY OF VORTEX MOTIONS IN COMPRESSIBLE STRATIFIED FLUIDS

INTRODUCTION

Vortex shedding behind an axisymmetric body submerged in stratified fluids has been puzzling many researchers because of the intrinsic vortex pattern developed in the late wake. When an axisymmetric body is towed through a stratified fluid at a particular Reynolds number, vortices are first shed three dimensionally. However, the gravitational effects induced by the density stratification soon inhibit the vertical motion. The resultant vortex structure is vertically oriented and resembles the two-dimensional Kármán vortex street behind a bluff body if observed from the gravitational direction. Even though the mechanism behind the development of this vortex structure is still not clear, the inhibition of the vertical motion can qualitatively be viewed by examining the motion of fluid particles in the gravitational force field. When a sphere or other axisymmetric body is towed through a stratified fluid, the fluid particles in four locations are of particular interest. Figures 1a and 1b respectively show the top view and side view of the sphere and of the four fluid particles A, B, C and D being considered. Since the fluid is stably stratified in the vertical direction, the densities of particles A and B are equal while the density of particle C is lighter than that of particle D. To create a rotational motion at the onset of shedding means that either particles A and B or particles C and D have to interchange their positions. The interchange of particles A and B requires no work done in the gravitational force field. The interchange of particles C and D, however, requires a work done equal to the increase of the potential energy at the new location. Accordingly, nature will take an easy way. With the vertical motion suppressed by the gravitational forces, the resultant vortex motion will be confined in a relatively two-dimensional motion reminiscent of the Kármán vortex street behind an infinitively long cylinder when viewed in the vertical direction.

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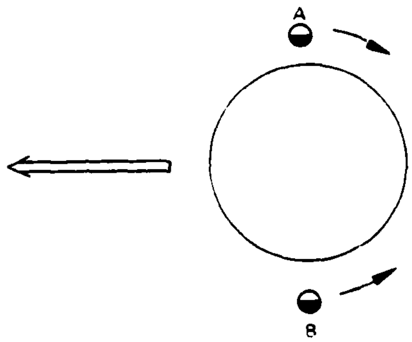

Fig. 1a — Top view of a sphere towed in a stably stratified fluid.

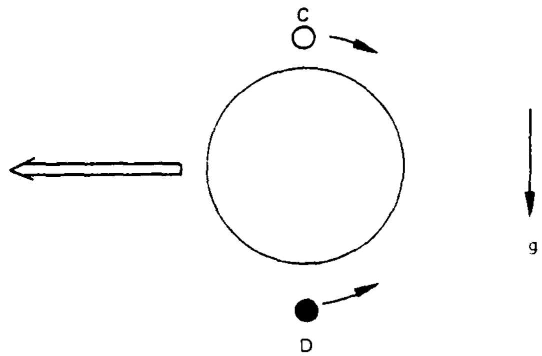

Fig. 1b — Side view of a sphere towed in a stably stratified fluid.

To understand the behavior of such vortex motions, we need to mathematically consider a more general type of flow profiles which can be used to describe this kind of flow behavior. Most of the steady profiles considered in parallel flows vary only in one direction, i.e., they are single coordinate-dependent, even though they may possess more than one independent velocity component. Relatively little attention has been given to flows varying in more than one direction. The vortex motion being considered is of double coordinate-dependency. A constraint relation, required to satisfy the pressure balance condition at all points within the flow domain, may play an important role in the characteristic of flow behavior. It will be seen such a constraint may be responsible for the variation or redistribution of the density and for the special vortex pattern developed in the stratified late wake behind a sphere. The effect on the density distribution by the presence of a pressure constraint can be qualitatively seen from the following discussion.

Consider a line vortex with its axis of symmetry coinciding with the z-axis of a cylindrical coordinate system (r, θ, z). The constraint equation for the present flow is

$$\frac{\partial}{\partial z}(\rho_o r \Omega^2) + \frac{\partial}{\partial r}(\rho_o g) = 0 \tag{1}$$

where ρ_o and Ω are respectively the density and the angular velocity of the flow. Let us first assume that the density is stratified only in the axial direction, i.e., $\rho_o = \rho_o(z)$. The constraint equation requires the angular velocity to be described by

$$\Omega = \omega(r)/\sqrt{\rho_o(z)} \tag{2}$$

where $\omega(r)$ is an arbitrary function of the radius. The steady-state pressure is now governed by

$$P(r,z) = \int r\omega^2(r)\,dr - g\int \rho_o(z)\,dz. \tag{3}$$

Now we consider two vortices, one on top of the other, with a common axis and a common boundary located at $z = Z$. The vortex on top has a density ρ_1 and an angular velocity Ω_1 while the one at the bottom has a density ρ_2 and an angular velocity Ω_2. The pressures in each individual region are, respectively,

$$\begin{aligned} P_1 &= \int r\omega_1^2\,dr - g\int \rho_1\,dz \\ P_2 &= \int r\omega_2^2\,dr - g\int \rho_2\,dz \end{aligned} \tag{4}$$

The pressure balance condition at the common boundary $z = Z$ requires that

$$P_1(r,Z) = P_2(r,Z)$$

This implies

$$\omega_1^2(r) = \omega_2^2(r)$$

or

$$\rho_1(Z)\,\Omega_1^2(r) = \rho_2(Z)\,\Omega_2^2(r) = \text{constant} \tag{5}$$

In other words, if the density were restricted to be z-axis dependent only, the angular velocity would be inversely proportional to the density of the fluid. Such a restriction in turn implies that, for a statically stable density distribution of a line vortex, the rotational velocity should be large on top and small at the bottom. This is apparently not the vortex pattern developed in the late wake behind a sphere. The above discussion suggests that, in order to generate the particular vortex pattern behind an axisymmetric body, the steady-state density will have to be redistributed from its original axis-dependent distribution. The density difference between the wake and the environment would be even sharper as a result of such a redistribution. This could be the cause for the existence of the particular vortex structure far down stream of the wake.

To understand the behavior of such a vortex pattern behind an axisymmetric body, one needs to investigate the stability characteristics of a general class of flows which have their density and velocity components varying in more than one direction.

A classical way to attack problems for flow stability is to apply the well-known normal mode method to a set of partial differential equations governing the stability characteristics of flows. The method is very effective for flow profiles with one independent variable since periodic solutions are readily admitted to the partial differential equations which then are reduced to a set of ordinary differential equations. For flow profiles with density and velocity distributions depending on more than one coordinate, the method of normal modes becomes exhausted because the partial differential equations governing flow stability in general can not be reduced to a corresponding set of ordinary differential equations. The method of separation of variables may provide a way to solve the problem, however, its application is limited to only a few special cases.

The method of generalized progressing wave expansion can circumvent this difficulty and provides us with a tool to attack stability problems for flows varying along more than one coordinate. In a recent paper by Eckhoff and Storesletten (1978), this method was applied to a class of inviscid helical gas flows with their steady-state profiles depending only on the radius. A necessary condition for stability was derived and compared with the existing stability criteria. In the present investigation, a general class of compressible vortex flows with their steady state distributions depending on the radial and axial coordinate are to be considered. Dissipation effects due to viscosity and thermal diffusivity are disregarded. Necessary conditions for stability of the flows are derived using the method of generalized progressing wave expansion. For the flows to be stable, it is necessary that they be stable in the centrifugal force field created by the rotation of the fluids, in the gravitational force field arisen from the density variation in the vertical direction, and in the pressure field balanced by centrifugal and gravitational forces. These conditions can be interpreted by a kinetic energy approach similar to the one used by Rayleigh (1916) and by a work done approach based on the movement of fluid particles in the centrifugal and gravitational force fields.

We have not been successful in obtaining a sufficient condition for stability. However, it is suggested, from the necessary condition to be derived, that a sufficiency condition for flows with the steady distributions varying in both the radial and axial directions should satisfy three criteria: one in the radial direction (Fung and Kurzweg, 1975; Lalas, 1975), one in the vertical direction (the classical Richardson criterion for 2D parallel flows), and a third one constrained by the pressure relation for the variation of the density and for the balance of the two force fields.

GOVERNING EQUATIONS

Consider an isentropic vortex motion having the axis of symmetry aligned with the z-axis of a cylindrical coordinate (r, θ, z) and with the direction of gravity. The fluid, rotating at an angular velocity $\Omega(r, z)$ and having a density $\rho_o(r, z)$, is assumed to be compressible but inviscid. To satisfy the pressure balance condition anywhere within the flow regime, the constraint equation (1) for the steady-state must be satisfied. Assume the perturbations to the steady-state flow profile are small. The linearized partial differential equations governing the system are

$$\begin{aligned}
&\rho_o \left\{ \frac{D\hat{u}}{Dt} - 2\Omega\,\hat{v} \right\} - r\,\Omega^2 \hat{\rho} = -\frac{\partial \hat{p}}{\partial r} \\
&\rho_o \left\{ \frac{D\hat{v}}{Dt} + \left\{ r\,\frac{\partial \Omega}{\partial r} + \Omega \right\} \hat{u} + r\,\frac{\partial \Omega}{\partial z}\,\hat{w} \right\} = -\frac{1}{r}\,\frac{\partial \hat{p}}{\partial \theta} \\
&\rho_o \left\{ \frac{D\hat{w}}{Dt} \right\} = -\frac{\partial \hat{p}}{\partial z} - g\hat{\rho} \\
&\frac{D\hat{\rho}}{Dt} + \frac{\partial \rho_o}{\partial r}\,\hat{u} + \frac{\partial \rho_o}{\partial z}\,\hat{w} = -\rho_o \left\{ \frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{1}{r}\,\frac{\partial \hat{v}}{\partial \theta} + \frac{\partial \hat{w}}{\partial z} \right\} \\
&\frac{D\hat{p}}{Dt} + \rho_o\,r\,\Omega^2 \hat{u} - \rho_o\,g\,\hat{w} = c_o^2 \left\{ \frac{D\hat{\rho}}{Dt} + \frac{\partial \rho_o}{\partial r}\,\hat{u} + \frac{\partial \rho_o}{\partial z}\,\hat{w} \right\}
\end{aligned} \tag{6}$$

where $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \Omega \dfrac{\partial}{\partial \theta}$ and c_o is the velocity of sound. The perturbations of velocity in r, θ and z directions are, respectively, $\hat{u}$, $\hat{v}$ and $\hat{w}$ while the perturbations of the density and pressure are, respectively, $\hat{\rho}$ and $\hat{p}$.

Consider the following transformation for the density and pressure perturbation such that

$$\hat{\rho} = \frac{\rho_o}{c_o} (\rho + p)$$
$$\hat{p} = c_o \rho_o p. \tag{7}$$

Equations (6) can be transformed into a symmetric hyperbolic system described by

$$\left\{ \frac{\partial}{\partial t} + \mathbf{A}_1 \frac{\partial}{\partial t} + \mathbf{A}_2 \frac{\partial}{\partial \theta} + \mathbf{A}_3 \frac{\partial}{\partial z} + \mathbf{B} \right\} \begin{vmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \rho \\ p \end{vmatrix} = 0 \tag{8}$$

where

$$\mathbf{A}_1 = \begin{vmatrix} 0 & 0 & 0 & 0 & c_o \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c_o & 0 & 0 & 0 & 0 \end{vmatrix} \quad \mathbf{A}_2 = \begin{vmatrix} \Omega & 0 & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 & c_o/r \\ 0 & 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & \Omega & 0 \\ 0 & c_o/r & 0 & 0 & \Omega \end{vmatrix}$$

$$\mathbf{A}_3 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_o \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_o & 0 & 0 \end{vmatrix}$$

$$\mathbf{B} = \begin{vmatrix} 0 & -2\Omega & 0 & -\dfrac{r\Omega^2}{c_o} & \dfrac{1}{\rho_o}\dfrac{\partial}{\partial r}(c_o\rho_o) - \dfrac{r\Omega^2}{c_o} \\ r\dfrac{\partial \Omega}{\partial r} + 2\Omega & 0 & r\dfrac{\partial \Omega}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{g}{c_o} & \dfrac{1}{\rho_o}\dfrac{\partial}{\partial r}(c_o\rho_o) + \dfrac{g}{c_o} \\ \dfrac{c_o}{\rho_o}\dfrac{\partial \rho_o}{\partial r} - \dfrac{r\Omega^2}{c_o} & 0 & \dfrac{c_o}{\rho_o}\dfrac{\partial \rho_o}{\partial z} + \dfrac{g}{c_o} & 0 & 0 \\ \dfrac{c_o}{r} + \dfrac{r\Omega^2}{c_o} & 0 & -\dfrac{g}{c_o} & 0 & 0 \end{vmatrix}$$

Following the method of generalized progressing wave expansion used by Eckhoff and Storesletten (1978) in their study on the stability of helical gas flows, we obtain the characteristic equation for the symmetric hyperbolic system described by (10) as follows:

$$\det |\xi_1 \mathbf{A}_1 + \xi_2 \mathbf{A}_2 + \xi_3 \mathbf{A}_3 - \lambda \mathbf{I}| = 0. \tag{9}$$

The characteristic roots of the determinant are given by

$$\begin{aligned} \lambda_1 &= \Omega \xi_2 \\ \lambda_2 &= \Omega \xi_2 + c_o \eta \\ \lambda_3 &= \Omega \xi_2 - c_o \eta. \end{aligned} \tag{10}$$

Here ξ_j are components for the orthonormal eigenvectors associated with the eigenvalue λ and

$$\eta = (\xi_1^2 + \xi_2^2/r^2 + \xi_3^2)^{1/2} \tag{11}$$

The eigenvalues λ_2 and λ_3 correspond to acoustic solutions and will not be discussed any further. The eigenvalue λ_1 corresponds to gravity waves and is the main concern in this paper. The corresponding ray equations for gravity waves are

$$\begin{aligned} \frac{dr}{dt} &= \frac{\partial \lambda_1}{\partial \xi_1} = 0 & \frac{d\xi_1}{dt} &= -\frac{\partial \lambda_1}{\partial r} = -\frac{\partial \Omega}{\partial r}\xi_2 \\ \frac{d\theta}{dt} &= \frac{\partial \lambda_1}{\partial \xi_2} = \Omega & \frac{d\xi_2}{dt} &= -\frac{\partial \lambda_2}{\partial \theta} = 0 \\ \frac{dz}{dt} &= \frac{\partial \lambda_1}{\partial \xi_3} = 0 & \frac{d\xi_3}{dt} &= -\frac{\partial \lambda_3}{\partial z} = -\frac{\partial \Omega}{\partial z}\xi_3 \end{aligned} \tag{12}$$

with the solutions to these ray equations given by

$$\begin{aligned} r &= r_o & \xi_1 &= \xi_{10} - \xi_{20} \left.\frac{\partial \Omega}{\partial r}\right|_{r_o, z_o} t \\ \theta &= \theta_o + \Omega(r_o, z_o)\, t & \xi_2 &= \xi_{20} \\ z &= z_o & \xi_3 &= \xi_{30} - \xi_{20} \left.\frac{\partial \Omega}{\partial z}\right|_{r_o, z_o} t \end{aligned} \tag{13}$$

where r_o, θ_o, z_o and ξ_{ko} $(k = 1,2,3)$ stand for the initial values at $t = 0$. The amplitude of the leading term of the generalized progressing wave expansion for the gravity waves is described by

$$L = \sum_{k=1}^{3} \sigma_k \mathbf{r}_k \tag{14}$$

where σ_k are scalar functions to be used as dependent variables in the transport equations, and r_k are orthonormal eigenvectors defined by

$$\mathbf{r}_l \cdot \mathbf{A}_k \mathbf{r}_m = \begin{cases} 0 & \text{for } l \neq m \\ \dfrac{\partial \lambda_1}{\partial \xi_k} & \text{for } l = m \end{cases} \quad k = 1,\ 2,\ 3. \tag{15}$$

The eigenvectors are selected such that

$$\begin{aligned} \mathbf{r}_1 &= \{\xi_2/r \quad \xi_3 \quad 0 \quad \xi_1 \quad 0\}/\eta \\ \mathbf{r}_2 &= \{0 \quad \xi_1 \quad -\xi_2/r \quad -\xi_3 \quad 0\}/\eta . \\ \mathbf{r}_3 &= \{-\xi_1 \quad 0 \quad -\xi_3 \quad \xi_2/r \quad 0\}/\eta \end{aligned} \tag{16}$$

The transport equations for the flow under consideration are given by

$$\frac{d\sigma_l}{dt} = -\sum_{k=1}^{3} \mathbf{r}_l \cdot \left\{\left\{\mathbf{A}_1 \frac{\partial \mathbf{r}_k}{\partial r} + B\mathbf{r}_k\right\}\right\} - \left\{\left\{\frac{\partial \lambda_1}{\partial r}\frac{\partial \mathbf{r}_k}{\partial \xi_1} + \frac{\partial \lambda_1}{\partial z}\frac{\partial \mathbf{r}_k}{\partial \xi_3}\right\}\right\}\sigma_k \quad {}_{l=1,2,3} \tag{17}$$

or

$$\frac{d\boldsymbol{\sigma}}{dt} = \mathbf{A}(t)\boldsymbol{\sigma}. \tag{18}$$

Here $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ and $\mathbf{A}(t)$ is a time dependent matrix with its elements described by

$$A_{lk} = a_{lk}/\eta^2. \tag{19}$$

Here also

$$a_{11} = -\left\{F_r - \frac{r\Omega^2}{c_o}\right\}\xi_1 \frac{\xi_2}{r} - D_\bullet(r\Omega)\frac{\xi_2}{r}\xi_3$$

$$a_{12} = \left[F_z + 2\Omega - \frac{\partial}{\partial z}(r\Omega)\right]\xi_1\frac{\xi_2}{r} + \left[-\frac{r\Omega^2}{c_o} + \frac{\partial}{\partial z}(r\Omega) + D_\bullet(r\Omega)\right]\frac{\xi_2}{r}\xi_3$$

$$a_{13} = F_r\xi_1^2 - \left[D_\bullet(r\Omega) - \frac{r\Omega^2}{c_o}\right]\frac{\xi_2^2}{r^2} + \frac{\partial}{\partial z}(r\Omega)\xi_3^2 + [F_z + D^*(r\Omega)]\xi_1\xi_3$$

$$a_{21} = -\left[D^*(r(\Omega) - \frac{\partial}{\partial z}(r\Omega) - \frac{g}{c_o}\right]\xi_1\frac{\xi_2}{r} + [F_r - D_\bullet(r\Omega)]\frac{\xi_2}{r}\xi_3$$

$$a_{22} = \frac{\partial}{\partial z}(r\Omega)\xi_1\frac{\xi_2}{r} - \left\{F_z + \frac{g}{c_o}\right\}\frac{\xi_2}{r}\xi_3$$

$$a_{23} = D^*(r\Omega)\xi_1^2 + \left[\frac{\partial}{\partial z}(r\Omega) + \frac{g}{c_o}\right]\frac{\xi_2^2}{r^2} - F_z\xi_3^2 - \left[F_r - \frac{\partial}{\partial z}(r\Omega)\right]\xi_1\xi_3$$

$$a_{31} = -\frac{r\Omega^2}{c_o}\xi_1^2 - [F_r - D_\bullet(r\Omega)]\frac{\xi_2^2}{r^2} - \left\{2\Omega - \frac{g}{c_o}\right\}\xi_1\xi_3$$

$$a_{32} = -2\Omega\,\xi_1^2 + \left[F_z - \frac{\partial}{\partial z}(r\Omega)\right]\frac{\xi_2^2}{r^2} - \frac{g}{c_o}\,\xi_3^2 + \frac{r\Omega^2}{c_o}\,\xi_1\xi_3$$

$$a_{33} = \left\{F_r - \frac{r\Omega^2}{c_o}\right\}\xi_1\frac{\xi_2}{r} + \left\{F_z + \frac{g}{c_o}\right\}\frac{\xi_2}{r}\,\xi_3$$

where

$$\begin{aligned} D^* &= \frac{\partial}{\partial r} + \frac{1}{r} \\ D_* &= \frac{\partial}{\partial r} - \frac{1}{r} \\ F_r &= \frac{c_o}{\rho_o}\frac{\partial \rho_o}{\partial r} - \frac{r\Omega^2}{c_o} \end{aligned} \tag{20}$$

and

$$F_z = \frac{c_o}{\rho_o}\frac{\partial \rho_o}{\partial z} + \frac{g}{c_o}.$$

The transport equations are valid along the rays described by Eqs. (13). The stability characteristics of the flow governed by partial differential Eqs. (6) are now equivalent to those of ordinary differential Eqs. (18) evaluated along the rays. The flow will not be stable unless the system (18) is stable for all possible choices of r_o, θ_o, z_o and ξ_{ko}.

STABILITY CHARACTERISTICS

The system described by the transport Eq. (18) is, in general, non-autonomous. A stable asymptotic behavior for large t will be required in order to have stability for the system. For $t \to \infty$, the non-vanishing elements of matrix **A** are

$$\begin{aligned} a_{13} &= \left\{F_r x^2 + [F_z + D^*(r\Omega)]\,xy + \frac{\partial}{\partial z}(r\Omega)\,y^2\right\}/\Delta \\ a_{23} &= \left\{D^*(r\Omega)\,x^2 - \left[F_r - \frac{\partial}{\partial z}(r\Omega)\right]xy - F_z y^2\right\}/\Delta \\ a_{31} &= \left\{-\frac{r\Omega^2}{c_o}x^2 - \left[2\Omega - \frac{g}{c_o}\right]xy\right\}/\Delta \\ a_{32} &= \left\{-2\Omega\,x^2 + \frac{r\Omega^2}{c_o}xy - \frac{g}{c_o}y^2\right\}/\Delta \end{aligned} \tag{21}$$

where $x = \dfrac{\partial \Omega}{\partial r}$, $y = \dfrac{\partial \Omega}{\partial z}$ and $\Delta = x^2 + y^2$. The characteristic equation for the eigenvalue λ of the non-autonomous system described by Eq. (21) can be found as

$$\lambda (\lambda^2 + \Psi/\Delta) = 0 \tag{22}$$

where

$$\Psi = \left\{F_r \frac{r\Omega^2}{c_o} + 2\Omega D^*(r\Omega)\right\} x^2 - \left\{F_r \frac{g}{c_o} - F_z \frac{r\Omega^2}{c_o} - 2\Omega \frac{\partial}{\partial z}(r\Omega)\right\} xy - F_z \frac{g}{c_o} y^2. \tag{23}$$

In order that the system with the elements described by Eqs. (21) shall be stable, it is necessary that all the characteristic roots in Eq. (22) have non-positive real parts, i.e.,

$$\Psi \geqslant 0. \tag{24}$$

Considering x and y as two independent variables, one may normalize condition (24) into

$$F_r \frac{r\Omega^2}{c_o} + 2\Omega D^*(r\Omega) - F_z \frac{g}{c_o} \pm$$
$$\left\{\left[F_r \frac{r\Omega^2}{c_o} + 2\Omega D^*(r\Omega) - F_z \frac{g}{c_o}\right]^2 + 4F_z \frac{g}{c_o}\left[F_r \frac{r\Omega^2}{c_o} + 2\Omega D^*(r\Omega)\right] + 4\left(F_r \frac{g}{c_o}\right)^2\right\}^{\frac{1}{2}} \geqslant 0. \tag{25}$$

Equation (25) will be satisfied if

$$F_r \frac{r\Omega^2}{c_o} + 2\Omega D^*(r\Omega) - F_z \frac{g}{c_o} \geqslant 0 \tag{26a}$$

and

$$-F_z \frac{g}{c_o}\left[F_r \frac{r\Omega^2}{c_o} + 2\Omega D^*(r\Omega)\right] \geqslant \left(F_r \frac{g}{c_o}\right)^2. \tag{26b}$$

Since the right hand side of Eq. (26b) is always positive, conditions (26) incorporated with Eqs. (20) require

$$\frac{r\Omega^2}{\rho_o}\frac{\partial \rho_o}{\partial r} + \phi - \left(\frac{r\Omega^2}{c_o}\right)^2 \geqslant 0 \tag{27a}$$

$$-\left[\frac{g}{\rho_o}\frac{\partial \rho_o}{\partial z} + \left(\frac{g}{c_o}\right)^2\right] \geqslant 0 \tag{27b}$$

and

$$-\left[\frac{g}{\rho_o}\frac{\partial \rho_o}{\partial z} + \left(\frac{g}{c_o}\right)^2\right]\left[\frac{r\Omega^2}{\rho_o}\frac{\partial \rho_o}{\partial r} + \phi - \left(\frac{r\Omega^2}{c_o}\right)^2\right] \geqslant \left(\frac{g}{r\Omega^2}\right)^2\left[\frac{r\Omega^2}{\rho_o}\frac{\partial \rho_o}{\partial r} - \left(\frac{r\Omega^2}{c_o}\right)^2\right]^2. \tag{27c}$$

Here $\phi = 2\Omega D^*(r\Omega)$ reduces to the classical Rayleigh discriminant if the azimuthal velocity of the flow is radius-dependent only.

For uniform rotation, i.e., Ω = constant, no criteria can be observed since Eq. (24) will become trivial. Special considerations will have to be taken in order to obtain a general stability criterion for every possible flow profile under the present assumption. For $\partial\Omega/\partial r = \partial\Omega/\partial z = 0$, system (18) becomes autonomous, and the ray equations (13) for autonomy reduce to

$$\xi_k = \xi_{ko} \quad k = 1, 2, 3 \tag{28}$$

The eigenvalues λ of system (18) are now governed by the equation

$$\lambda\,[\lambda^2 + \psi/(\xi_{10}^2 + \xi_{20}^2 + \xi_{30}^2)] = 0 \tag{29}$$

where

$$\psi = \left(F_r \frac{r\Omega^2}{c_o} + 4\Omega^2\right)\xi_{10}^2 - \left(F_r \frac{g}{c_o} - F_z \frac{r\Omega^2}{c_o}\right)\xi_{10}\xi_{30} - F_z \frac{g}{c_o}\xi_{30}^2 + \left(F_r \frac{r\Omega^2}{c_o} - F_z \frac{g}{c_o}\right)\left(\frac{\xi_{20}}{r}\right)^2$$

In order to have system (18) be autonomous and stable, it is required that the eigenvalues have non-positive real parts, i.e.,

$$\psi \geqslant 0. \tag{30}$$

Since ξ_{10}, ξ_{20} and ξ_{30} are independent components of the orthonormal eigenvectors, inequality (30) for arbitrary values of ξ_{10}, ξ_{20} and ξ_{30} is equivalent to

$$F_r \frac{r\Omega^2}{c_o} - F_z \frac{g}{c_o} \geqslant 0 \tag{31a}$$

and

$$\left(F_r \frac{r\Omega^2}{c_o} + 4\Omega^2\right)\xi_{10}^2 - \left(F_r \frac{g}{c_o} - F_z \frac{r\Omega^2}{c_o}\right)\xi_{10}\xi_{30} - F_z \frac{g}{c_o}\xi_{30}^2 \geqslant 0 \tag{31b}$$

Equation (31b), as a quadratic equation for arbitrary values of ξ_{10} and ξ_{30}, can be normalized into the following two inequalities

$$\left(F_r \frac{r\Omega^2}{c_o} + 4\Omega^2\right) - F_z \frac{g}{c_o} \geqslant 0 \tag{32a}$$

$$F_r \frac{r\Omega^2}{c_o} = -\left(\frac{r\Omega^2}{g}\right)^2\left(F_z \frac{g}{c_o}\right) \geqslant 0. \tag{32b}$$

It can be seen immediately Eqs. (31a) and (32a) will be satisfied automatically if Eq. (32b) is satisfied. To have stability of the system of equations (18) for autonomy, it is therefore required that

$$\frac{r\Omega^2}{\rho_o}\frac{\partial\rho_o}{\partial r} - \left(\frac{r\Omega^2}{c_o}\right)^2 \geqslant 0 \tag{33a}$$

$$-\left[\frac{g}{\rho_o}\frac{\partial\rho_o}{\partial z} + \left(\frac{g}{c_o}\right)^2\right] \geqslant 0. \tag{33b}$$

Equations (27) and (33), evaluated along the rays governed by Eqs. (13), represent the stability conditions of the system of equations (18) for both non-autonomous and autonomous cases, respectively. The stability characteristics of the flow governed by the partial differential Eq (6) are equivalent to those of the system governed by the ordinary differential Eq. (18) evaluated along the rays. It can therefore be concluded that the necessary conditions for stability of the compressible stratified flows under consideration require the inequalities

$$N_r^2 + \phi \geqslant 0 \tag{34a}$$

$$N_z^2 \geqslant 0 \tag{34b}$$

$$(N_r^2 + \phi)N_z^2 \geqslant N_r^4/F^2 \tag{34c}$$

to be satisfied anywhere within the flow regime. Here the Brunt-Värsälä frequencies are defined as

$$N_r^2 = \frac{r\Omega^2}{\rho_o}\frac{\partial\rho_o}{\partial r} - \left(\frac{r\Omega^2}{c_o}\right)^2 \tag{35a}$$

$$N_z^2 = -\frac{g}{\rho_o}\frac{\partial\rho_o}{\partial z} - \left(\frac{g}{c_o}\right)^2 \tag{35b}$$

to respectively represent the density variations along the radial and axial force fields. The Froude number is defined as

$$F = \frac{r\Omega^2}{g} \tag{36}$$

to represent the ratio between the centrifugal and gravitational forces. Equation (34a) will be replaced by $N_r^2 \geqslant 0$ for uniformly rotating flows. It is worth mentioning that Eqs. (34) actually do not represent three independent conditions. Condition (34c) implies that either one of the two conditions in (34a) and (34b) must be automatically satisfied if the other one is satisfied. This behavior as well as the physical mechanism behind the three conditions will be explained in the following section.

INTERPRETATION OF THE RESULTS

The necessary conditions for stability previously derived can be explained based on a kinetic energy approach similar to the one used by Rayleigh (1916) and on a work done approach based on the movement of fluid particles in the centrifugal and gravitational force fields. For simplicity, compressibility effects, in general destabilizing the flow as implied by conditions (34), will be ignored in the following explanation of the stability conditions. It should be first pointed out that, for the inviscid flow under consideration, the principle of conservation of circulation is valid, since the density is single-valued. This is the key point for the physical arguments of the stability condition to be immediately discussed.

Consider two fluid particles originally located at Q_1 and Q_2 within the flow regime in the r-z plane as illustrated in Fig. 2. The particle at Q_1 has a density ρ_o and a velocity v_o while the particle at Q_2 has a density $\rho_o + \delta\rho_o$ and a velocity $v_o + \delta v_o$. Here $v_o = r\Omega$ and $\delta = \delta r \dfrac{\partial}{\partial r} + \delta z \dfrac{\partial}{\partial z}$. To explain the physical meaning of the necessary conditions for stability, we first use the energy approach by considering the variation of the total energy as a result of a perturbation to the system. In the steady-state, the kinetic and potential energy of the two particles are given by

$$\text{K.E.} \qquad \frac{1}{2}\{\rho_o v_o^2 + (\rho_o + \delta\rho_o)(v_o + \delta v_o)^2\} \tag{37}$$

and

$$\text{P.E.} \qquad \rho_o gz + (\rho_o + \delta\rho_o)\, g(z + \delta z).$$

When the two particles interchange their positions as a result of a perturbation, the kinetic and the potential energy of the perturbed system are

$$\text{K.E.} \qquad \frac{1}{2}\left\{\rho_o\left[\frac{r v_o}{r + \delta r}\right]^2 + (\rho_o + \delta\rho_o)\left[\frac{(r + \delta r)(v_o + \delta v_o)}{r}\right]^2\right\}$$

and

$$\text{P.E.} \qquad \rho_o g(z + \delta z) + (\rho_o + \delta\rho_o) gz. \tag{38}$$

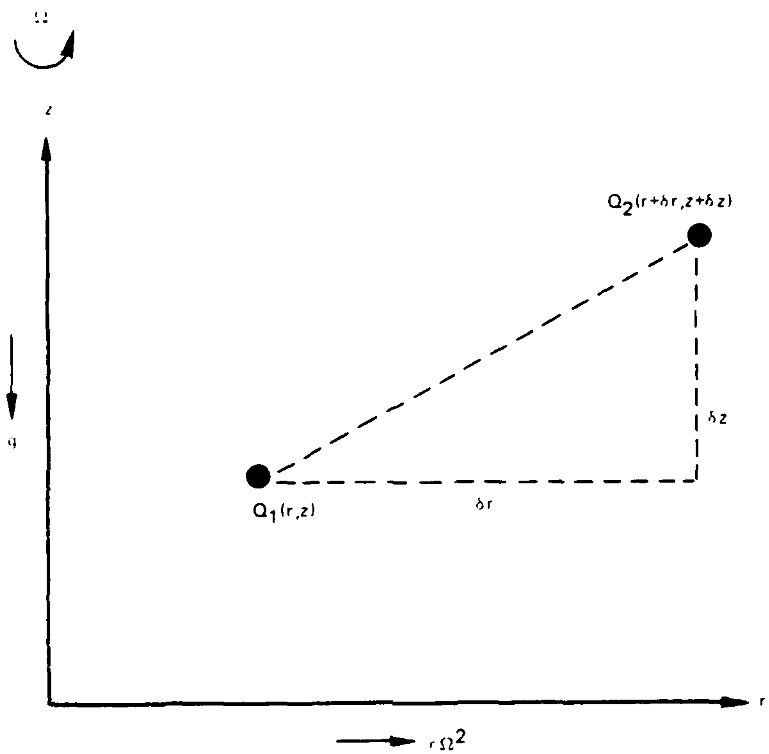

Fig. 2 — Coordinate of the fluid particles

Here the conservation of circulation has been applied to the resultant kinetic energy in (38). If the perturbation is small, the stability of the system requires that the leading term of the energy variation as a result of interchange of the two particles be equal or greater than zero, i.e.,

$$\left[\frac{r\Omega^2}{\rho_o}\frac{\partial\rho_o}{\partial r}+\phi\right](\delta r)^2-\left[\frac{g}{\rho_o}\frac{\partial\rho_o}{\partial r}-\frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o r\Omega^2)\right](\delta r)(\delta z)-\frac{g}{\rho_o}\frac{\partial\rho_o}{\partial z}(\delta z)^2\geqslant 0 \qquad (39)$$

everywhere within the flow domain.

An alternative approach one may use to observe the stability characteristics of the system is to examine the work done by the two particles in the centrifugal and gravitational force fields. When the two particles interchange their positions, the work done by the particle originally located at Q_1 is

$$W_1=-\frac{\rho_o}{2}\left\{\frac{v_o^2}{r}+\frac{(r\,v_o)^2}{(r+\delta r)^3}\right\}\delta r+\rho_o g\,\delta z \qquad (40)$$

while the work done by the particle originally located at Q_2 is

$$W_2 = \frac{\rho_o + \delta\rho_o}{2}\left\{\frac{(v_o + \delta v_o)^2}{(r + \delta r)} + \frac{[(r + \delta r)(v_o + \delta v_o)]^2}{r^3}\right\}\delta r - (\rho_o + \delta\rho_o)\, g\,\delta z. \tag{41}$$

Condition (39) can be reached following the argument that the stability of the system requires the leading terms of the total work done by the interchange of the two particles to be non-negative, i.e.,

$$W_1 + W_2 = \frac{1}{r^3}\delta\,[\rho_o r^2 v_o^2]\,\delta r + \delta(\rho_o g)\delta z \geqslant 0 \tag{42}$$

The physical meaning of the necessary condition for stability derived by the method of generalized progressing wave expansion can be seen immediately if one compares Eq. (24) with Eq. (39) or (42). Equation (24) for incompressible fluids reduces to

$$\left[\frac{r\Omega^2}{\rho_o}\frac{\partial\rho_o}{\partial r} + \phi\right]\left(\frac{\partial\Omega}{\partial r}\right)^2 - \left[\frac{g}{\rho_o}\frac{\partial\rho_o}{\partial r} - \frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o r\Omega^2)\right]\left(\frac{\partial\Omega}{\partial r}\right)\left(\frac{\partial\Omega}{\partial z}\right) - \left[\frac{g}{\rho_o}\frac{\partial\rho_o}{\partial z}\right]\left(\frac{\partial\Omega}{\partial z}\right)^2 \geqslant 0. \tag{43}$$

Equations (39) and (43) are identical if one considers $\frac{\partial\Omega}{\partial r}$, $\frac{\partial\Omega}{\partial z}$, δr and δz as independent variables and match $\frac{\partial\Omega}{\partial r}$ with δr and $\frac{\partial\Omega}{\partial z}$ with δz. If the variation in the axial direction is restrained, Eqs. (39) and (43) reduce to the well-known Rayleigh-Synge criterion (Synge 1933) which is a requirement for centrifugal stability. Also it can be easily shown, by the same integral method used by Fung (1980), that the first term in Eqs. (39) or (43) is in fact a differential representation of a stable centrifugal force field. This mechanism is reflected in Eq. (34a) saying that the steady state of the flow should be stable in the radial direction. Two parts are involved in this first term. The first part is the variation of density in the centrifugal force field. The second part is the Rayleigh discriminant representing a variation of the centrifugal balance as a result constrained by the conservation of circulation.

If the variation in the radial direction is suppressed, Eqs. (39) and (43) reduce to a condition representing the variation of density in the gravitational force field. This mechanism is reflected in (34b) saying that the steady state of density should be statically stable in the axial direction.

The second term in Eqs. (39) and (43) represents the cross correlation between the radial and axial variations of the density in the centrifugal and gravitational force fields. This variation is reflected in condition (34c) with a mechanism to be immediately revealed. For incompressible flows, Eq. (34c) reduces to

$$-\left[1+\frac{\phi}{\dfrac{r\Omega^2}{\rho_o}\dfrac{\partial\rho_o}{\partial r}}\right]\frac{\dfrac{\partial\rho_o}{\partial z}}{\dfrac{\partial\rho_o}{\partial r}} \geqslant \frac{1}{F}. \tag{44}$$

Equation (44) represents a requirement for stability imposed on the density variations in the radial and axial directions. The second term inside the bracket in the above equation is the Rayleigh discriminant normalized by the centrifugal force as a result of the density variation in the radial direction. It is a ratio of the variation of centrifugal forces between that resulting from the conservation of circulation and that arising from the variation of density in the centrifugal force field. For fixed values of the Froude number, the density variation in the radial direction, as compared to that in the axial direction, plays an opposite role in flow stability. While large density gradients (positive) in the axial direction stabilize the flow as in the case of two dimensional stratified fluids in a gravitational force field, large density gradients (positive) in the radial direction destabilize the flow. This is opposite to the role played by density gradients in radius-dependent rotating flows, and seems to be implausible at the first look. However, this is also an interesting point that will reveal the physical mechanism carried by Eqs. (34c) and (44). For potential flows, both Eqs. (1) and (44) reduce to

$$-\frac{\partial\rho_o}{\partial r}\Big/\frac{\partial\rho_o}{\partial z} = F = \frac{r\Omega^2}{g} \tag{45}$$

saying that the ratio between the density gradient in the radial direction and that in the axial direction should be compatible with the ratio between the centrifugal force field and the gravitational force field. And it is also the constraint condition for the pressure balance requires that large density gradients (positive) in the radial direction result in small density gradients in the axial direction, destabilizing the flow.

Based on the arguments just presented for the physical mechanisms of the necessary conditions, we conclude that Eqs. (34) represent a generalized state of statically stable profiles for the steady flow. To secure stability for the basic flow, it is necessary that the steady-state distribution satisfy the radial force balance condition, the axial force balance condition, and a pressure balance condition constraining the variations of density in both the centrifugal and gravitational force fields. As a result of the third constraint, Eqs. (34) do not represent three independent conditions. Either one of the two conditions in (34a) and (34b) will have to be automatically satisfied if the other condition and Eq. (34c) are fulfilled.

CONCLUSIONS AND DISCUSSIONS

Necessary conditions of stability for an isentropic compressible vortex flow, with the density being stratified in both the radial and axial directions, have been derived using the method of generalized progressing wave expansion. This method transforms a set of partial differential equations into a set of ordinary differential equations with equivalent stability characteristics evaluated along the rays. Thus it provides one with a powerful tool to attack a class of steady flows with more than one independent variable, in which the method of normal modes is exhausted. The necessary conditions derived here represent a generalized state of statically stable distribution for the steady flow. They require the flow to satisfy the centrifugal force balance condition, the gravitational force balance condition, and the pressure balance condition restraining the variation of densities and forces in both force fields.

It is shown in the Appendix, by using the Buckingham π theorem, that the general class of vortex flows being considered possesses three non-dimensional numbers. They are the Mach number, the Froude number, and the Richardson number (the Reynolds number will appear if viscous effects are considered). The first two are brought out by the necessary conditions. The last one is obviously embedded in a sufficiency condition for stability yet to be derived. One way to approach such a sufficiency condition is to construct a Liapunov function for the system of equation (14) using Liapunov's direct method. We have not been successful in such an approach.

Even though it has not been proved, several implications concerning the sufficiency condition or a Richardson criterion can be observed from the necessary conditions. In view of Eqs. (34), the sufficiency condition should also consist of three criteria: one in the radial direction (Fung & Kurzweg, 1975; Lalas, 1975), one in the vertical direction (the classical Richardson criterion for 2D parallel flows), and a third one imposed by the pressure constraint equation. Such a conjecture is based on the necessary and sufficient conditions for stability so far derived for two-dimensional stratified parallel flows and radially stratified rotating flows. For two-dimensional incompressible flows with densities stratified in z-direction, the necessary condition for stability, or "statically stable" as usually called, is

$$-\frac{d\rho_o}{dz} \geqslant 0 \tag{46}$$

the corresponding sufficiency condition is

$$\frac{-g\dfrac{d\rho_o}{dz}}{\rho_o\left(\dfrac{dU}{dz}\right)^2} \geqslant \frac{1}{4} \tag{47}$$

where U is the parallel velocity perpendicular to the direction of gravity. For incompressible rotating flows with densities and velocities varying only in the radial direction, the necessary condition for stability is

$$\Phi = \frac{r\Omega^2}{\rho_o}\frac{d\rho_o}{dr} + \phi \geqslant 0 \tag{48a}$$

or

$$\frac{d\rho_o}{dr} \geqslant 0 \tag{48b}$$

for uniformly rotating flows. Here Φ is the well-known Rayleigh-Synge discriminant and Eq. (48a) becomes the necessary and sufficient condition for stability if only axisymmetric disturbances are considered. The corresponding sufficient conditions for stability of the flow subject to azimuthal disturbances is

$$\frac{\Phi}{(rD\Omega + r\Omega)^2} \geqslant \frac{1}{4} \tag{49a}$$

or in an alternative form

$$\frac{r\Omega^2\dfrac{d\rho_o}{dr}}{\rho_o(rD\Omega)^2} \geqslant \frac{1}{4} \tag{49b}$$

Based on conditions (30) and on the stability criteria just discussed, we may propose the following sufficient conditions for stability of the general class of flows being considered.

If the Brunt-Väisälä frequencies are defined as in Eqs. (35) to respectively represent the density variations in the radial and axial directions, the appropriate local Richardson numbers in the corresponding direction may be defined as:

$$J_r = \frac{N_r^2 + \phi}{\left[r \dfrac{\partial \Omega}{\partial r} + 4\Omega \right]^2} \tag{50}$$

and

$$J_z = \frac{N_z^2}{\left[r \dfrac{\partial \Omega}{\partial z} \right]^2} \tag{51}$$

The plausible sufficiency conditions for a system subject to azimuthal disturbances, if proved, would have the forms

$$J_r \geqslant \frac{1}{4} \tag{52a}$$

$$J_z \geqslant \frac{1}{4} \tag{52b}$$

$$f(J_r, J_z) \geqslant 0 \tag{52c}$$

where $f(J_r, J_z)$ is an arbitrary function of the two Richardson numbers. Equation (52a) is a Richardson criterion in the radial direction and can also be written as

$$\frac{N_r^2}{\left[r \dfrac{\partial \Omega}{\partial r} \right]^2} \geqslant \frac{1}{4} \tag{53}$$

Equation (52b) is a Richardson criterion in the axial direction while Eq. (52c) is a pressure balance requirement constraining the two Richardson criteria in both directions. As implied by the necessary conditions for stability in Eqs. (34), the criteria described by Eqs. (52) would not represent three independent conditions. Either one of the two conditions in (52a) or (52b) would automatically be satisfied if the other condition and Eq. (52c) are fulfilled. Furthermore, the necessary conditions will serve as a prerequisite for the derivation of the sufficiency conditions yet to be identified.

An understanding of the stability criteria for the flow under consideration is essential to the vortex motion in the late wake behind a towed axisymmetric body. The existence of the vertically oriented vortex structure far downstream of the wake implies that the vortex motion satisfies stability criteria such as those given by Eqs. (52). The stability criteria, if derived, will provide us with an insight into the flow characteristics that may be used to predict the evolution and breakdown of the vertically oriented vortex motion.

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APPENDIX

The dimensionless numbers for inviscid and compressible vortex motion in the stratified wake behind an axisymmetric body are derived as follows. The independent dimensions for the flows are:

M = mass

L = characteristic length

T = time.

The independent variables are:

ρ_o = density (M/L^3)

U = velocity (L/T)

D = characteristic length of the body (L)

g = gravitational constant (M/TL)

c_o = velocity of sound (L/T)

N^2 = Brunt-Väisälä frequency $(1/T^2)$

The Buckingham π theorem states that the number of independent dimensionless numbers is equal to the number of independent variables less the number of independent dimensions, i.e., $6 - 3 = 3$. These three independent dimensionless numbers are constructed in the following way:

Dimensionless number

$$\pi_1 = \rho_o^{\alpha_1} U^{\alpha_2} D^{\alpha_3} c_o$$

$$= \left(\frac{M}{L}\right)^{\alpha_1} \left(\frac{L}{T}\right)^{\alpha_2} (L)^{\alpha_3} \left(\frac{L}{T}\right)$$

$$= M^{\alpha_1} L^{-3\alpha_1+\alpha_2+\alpha_3+1} T^{-\alpha_2-1}$$

Therefore

$$\alpha_1 = 0$$

$$\alpha_2 = -1$$

$$\alpha_3 = 0$$

and

$$\pi_1 = \rho_o^0 U^{-1} D^0 c_o = \frac{1}{\mathrm{Ma}}$$

where

$$\text{Mach number Ma} \triangleq \frac{U}{c_o}.$$

Dimensionless number
$$\begin{aligned}\pi_2 &= \rho_o^{\beta_1} U^{\beta_2} D^{\beta_3} g \\ &= \left(\frac{M}{L^3}\right)^{\beta_1} \left(\frac{L}{T}\right)^{\beta_2} (L)^{\beta_3} \left(\frac{L}{T^2}\right) \\ &= M^{\beta_1} L^{-3\beta_1+\beta_2+\beta_3+1} T^{-\beta_2-2}.\end{aligned}$$

Therefore

$$\beta_1 = 0$$
$$\beta_2 = -2$$
$$\beta_3 = 1$$

and

$$\pi_2 = \rho_o^0 U^{-2} D^1 g = \frac{1}{F}$$

where

$$\text{Froude number F} \triangleq \frac{U^2}{Dg}.$$

Dimensionless number
$$\begin{aligned}\pi_3 &= \rho_o^{\gamma_1} U^{\gamma_2} D^{\gamma_3} N^2 \\ &= \left(\frac{M}{L^3}\right)^{\gamma_1} \left(\frac{L}{T}\right)^{\gamma_2} (L)^{\gamma_3} \left(\frac{1}{T^2}\right) \\ &= M^{\gamma_1} L^{-3\gamma_1+\gamma_2+\gamma_3} T^{-\gamma_2-2}.\end{aligned}$$

Therefore

$$\gamma_1 = 0$$
$$\gamma_2 = -2$$
$$\gamma_3 = 2$$

and

$$\pi_3 = \rho_o^0 U^{-2} D^2 N^2 = J$$

where

$$\text{Richardson number } J \triangleq \frac{D^2 N^2}{U^2}.$$